

BOUNDARY AMENABILITY OF GROUPS VIA ULTRAPOWERS

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ABSTRACT. We use C^* -algebra ultrapowers to give a new construction of the Stone-Cech compactification of a separable, locally compact space. We use this construction to give a new proof of the fact that groups that act isometrically, properly, and transitively on trees are boundary amenable.

1. INTRODUCTION

Suppose that the discrete group Γ acts continuously on a compact space X . We say that the action of Γ on X is *amenable* if there is a net of continuous functions $x \mapsto \mu_n^x : X \rightarrow P(\Gamma)$ such that, for all $\gamma \in \Gamma$, we have

$$\sup_{x \in X} \|\gamma \cdot \mu_n^x - \mu_n^{\gamma \cdot x}\|_1 \rightarrow 0.$$

We say that Γ is *boundary amenable* if Γ acts amenably on some compact space. Note that amenable groups are precisely the groups that act amenably on a one-point space, whence they are boundary amenable. A prototypical example of a boundary amenable group that is not amenable is any non-abelian finitely generated free group. Boundary amenable groups are sometimes referred to as exact groups for the reduced group C^* -algebra $C_r^*(\Gamma)$ is exact (meaning that the functor $\otimes_{\min} C_r^*(\Gamma)$ is exact) if and only if Γ is boundary amenable.

In this note, we show how one can construct the Stone-Cech compactification of a separable, locally compact space using C^* -algebra ultrapowers. When applied to the case of a tree, this construction gives a very natural proof of the fact that a group that acts isometrically, properly, and transitively on a tree is boundary amenable. It was our initial hope that this construction could be used to settle the boundary amenability of groups where the answer was unknown (most notably *Thompson's group*) but we have thus far been unsuccessful (although remain optimistic). The naïve idea behind our optimism is that groups such as Thompson's group “almost” act isometrically on a tree and it is often the case that ultrapower constructions can turn almost phenomena into exact ones.

In Section 2, we explain the needed background on groups acting on C^* -algebras as well as ultrapowers of C^* -algebras. In Section 3, we explain our main construction in the general setting of separable, locally compact

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spaces. Finally, in Section 4 we use our construction to prove the boundary amenability of groups acting isometrically, properly, and transitively on trees.

2. PRELIMINARIES

2.1. Boundary amenability of groups acting on C^* -algebras. We will verify that certain groups act amenably on a compact space by checking that the group acts amenably on a unital abelian C^* -algebra as we now explain. Suppose that B is a unital C^* -algebra and that Γ acts on B . We consider the space $C_c(\Gamma, B)$ of finitely supported functions $\Gamma \rightarrow B$. $C_c(\Gamma, B)$ is naturally a $*$ -algebra with respect to the convolution product

$$(f * g)(\gamma) = \sum_{\gamma_1 \cdot \gamma_2 = \gamma} f(\gamma_1)(\gamma_1 \cdot g(\gamma_2))$$

and involution

$$f^*(\gamma) = \gamma \cdot f(\gamma^{-1})^*.$$

We also view $C_c(\Gamma, B)$ as a pre-Hilbert B -module with B -valued inner product $\langle f, g \rangle_B = \sum_{\gamma \in \Gamma} f(\gamma)^* g(\gamma)$ and corresponding norm $\|f\|_B := \|\langle f, f \rangle_B\|^{-1/2}$.

Recall also that an action of Γ on a compact space X induces an action of Γ on $C(X)$ by $(\gamma \cdot f)(x) := f(\gamma^{-1}x)$.

Our approach to showing that groups are boundary amenable is via the following reformulation of amenable actions (see [2, Proposition 2.2]).

Fact 2.1. *The action $G \curvearrowright X$ is amenable if and only if there exists a net $T_i \in C_c(G, C(X))$ such that, for each $\gamma \in \Gamma$ and each i , we have:*

- (1) $T_i(\gamma) \geq 0$;
- (2) $\langle T_i, T_i \rangle_{C(X)} = 1$;
- (3) $\|T_i - \delta_\gamma * T_i\|_{C(X)} \rightarrow 0$.

In our proofs below, we will have an action of a group Γ on a unital, abelian C^* -algebra B and we will prove that there exist $T_i \in C_c(\Gamma, B)$ satisfying the clauses (1)-(3) in the aforementioned fact. By Gelfand theory, B is isomorphic to $C(X)$ for some compact space X . It remains to observe that Gelfand theory respects the group action, meaning that we obtain an induced action of Γ on X such that the corresponding action of Γ on $C(X)$ “is” the corresponding action of Γ on B . Thus, our criterion for boundary amenability of a group is the following:

Fact 2.2. *A group Γ is boundary amenable if and only if there is a unital, abelian C^* -algebra B and a net $T_i \in C_c(\Gamma, B)$ such that, for each $\gamma \in \Gamma$ and each i , we have:*

- (1) $T_i(\gamma) \geq 0$;
- (2) $\langle T_i, T_i \rangle_B = 1$;
- (3) $\|T_i - \delta_\gamma * T_i\|_B \rightarrow 0$.

2.2. Ultrapowers of C^* algebras. Recall that a nonprincipal ultrafilter \mathcal{U} on \mathbb{N} is a $\{0, 1\}$ -valued measure on *all* subsets of \mathbb{N} such that finite sets get measure 0. We usually identify a nonprincipal ultrafilter with its collection of measure 1 sets, whence we write $A \in \mathcal{U}$ to indicate that A has measure 1. If $P(n)$ is a property of natural numbers, we say that $P(n)$ holds \mathcal{U} -almost everywhere if the set of n for which $P(n)$ holds belongs to \mathcal{U} . If (r_n) is a bounded sequence of real numbers, then the *ultralimit of (r_n) with respect to \mathcal{U}* , denoted $\lim_{n, \mathcal{U}} r_n$ or even $\lim_{\mathcal{U}} r_n$, is the unique real number r such that, for every $\epsilon > 0$, we have $|r_n - r| < \epsilon$ \mathcal{U} -almost everywhere.

Suppose that A is a unital C^* -algebra and \mathcal{U} is a nonprincipal ultrafilter on \mathbb{N} . We can define a seminorm $\|\cdot\|_{\mathcal{U}}$ on $\ell^\infty(A)$ by setting $\|(f_n)\|_{\mathcal{U}} := \lim_{\mathcal{U}} \|f_n\|$. We set $A^{\mathcal{U}}$ to be the quotient of $\ell^\infty(A)$ by those elements of $\|\cdot\|_{\mathcal{U}}$ -norm 0; we refer to $A^{\mathcal{U}}$ as the *ultrapower of A with respect to \mathcal{U}* . It is well known that $A^{\mathcal{U}}$ is once again a unital C^* -algebra. For $(f_n) \in \ell^\infty(A)$, we let $(f_n)^\bullet$ denote its image in $A^{\mathcal{U}}$. The canonical *diagonal embedding* $\Delta : A \rightarrow A^{\mathcal{U}}$ is given by $\Delta(a) = (a)^\bullet$.

3. THE MAIN CONSTRUCTION

In this section, we consider a second countable, locally compact space X with fixed basepoint $o \in X$. It is well-known that X admits a compatible proper metric d (see [4, Theorem 2]), and we fix such a metric in the rest of this section. For $r \in \mathbb{R}^{>0}$, we set $B(r)$ to be the closed ball of radius r around o .

We set $A = C_o(X)$, the space of complex-valued continuous functions on X that vanish at infinity. For $(f_n) \in \ell^\infty(A)$, we say that (f_n) is *\mathcal{U} -equicontinuous on bounded sets* if, for every $r, \epsilon > 0$, there is $\delta > 0$ such that, for \mathcal{U} -many n , we have for all $s, t \in B(r)$ with $d(s, t) < \delta$, that $|f_n(s) - f_n(t)| \leq \epsilon$.

Given any $(f_n) \in \ell^\infty(A)$, set $f_{\mathcal{U}} : X \rightarrow \mathbb{C}$ by $f_{\mathcal{U}}(t) := \lim_{\mathcal{U}} f_n(t)$. Note that $f_{\mathcal{U}}$ is a bounded function. The following lemma is quite routine and left to the reader.

Lemma 3.1. *If (f_n) is \mathcal{U} -equicontinuous on bounded sets, then $f_{\mathcal{U}}$ is uniformly continuous on bounded sets.*

Lemma 3.2. *Suppose that $(f_n)^\bullet = (g_n)^\bullet$ and (f_n) is \mathcal{U} -equicontinuous on bounded sets. Then so is (g_n) .*

Proof. Fix $r, \epsilon > 0$. Take $\delta > 0$ that witnesses \mathcal{U} -equicontinuity of (f_n) on $B(r)$ for $\epsilon/3$. Then for \mathcal{U} -many n , we have, for $s, t \in B(r)$ with $d(s, t) < \delta$, that

$$|g_n(s) - g_n(t)| \leq 2\|g_n - f_n\| + |f_n(s) - f_n(t)| \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

The previous lemma allows us to consider the *continuous part* of $A^{\mathcal{U}}$

$$A^{\mathcal{d}\mathcal{U}} := \{(f_n)^\bullet \in A^{\mathcal{U}} : (f_n) \text{ is } \mathcal{U}\text{-equicontinuous on bounded sets}\}.$$

Lemma 3.3.

- (1) $A^{\mathcal{d}\mathcal{U}}$ is a C^* -subalgebra of $A^{\mathcal{U}}$.
- (2) $\Delta(A) \subseteq A^{\mathcal{d}\mathcal{U}}$.

Proof. For (1), it is clear that $A^{\mathcal{d}\mathcal{U}}$ is a $*$ -subalgebra of $A^{\mathcal{U}}$. We must show that $A^{\mathcal{d}\mathcal{U}}$ is closed in $A^{\mathcal{U}}$. Towards this end, suppose that $(f_n^m)^\bullet$ is a sequence in $A^{\mathcal{d}\mathcal{U}}$ such that $(f_n^m)^\bullet \rightarrow (h_n)^\bullet$ as $n \rightarrow \infty$. We need to show that $(h_n)^\bullet \in A^{\mathcal{d}\mathcal{U}}$. Fix $r, \epsilon > 0$. Choose $m \in \mathbb{N}$ such that $\|(f_n^m)^\bullet - (h_n)^\bullet\| < \epsilon/3$. Suppose that $s, t \in B(r)$ are such that $d(s, t) < \delta$. Then we have that, for \mathcal{U} -many n , that

$$|h_n(s) - h_n(t)| \leq 2\|f_n^m - h_n\| + |f_n^m(s) - f_n^m(t)| \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

(2) follows from the fact that balls $B(r)$ in X are compact, whence elements of A are uniformly continuous on such balls. \square

We now consider

$$I := \{(f_n)^\bullet \in A^{\mathcal{U}} : (\exists r_n \in \mathbb{R})(\lim_{\mathcal{U}} r_n = +\infty \text{ and } f_n|_{B(o, r_n)} \equiv 0)\}.$$

It is clear from the definition that $I \subseteq A^{\mathcal{d}\mathcal{U}}$.

In the rest of this section, we fix continuous functions $\chi_n : X \rightarrow \mathbb{R}$ such that:

- (1) $0 \leq \chi_n \leq 1$;
- (2) $\chi_n(t) = 0$ for $t \in B(n)$;
- (3) $\chi_n(t) = 1$ when $d(t, o) \geq n + 1$.

Proposition 3.4.

- (1) I is a closed ideal in $A^{\mathcal{U}}$.
- (2) $A^{\mathcal{d}\mathcal{U}}/I$ is unital.
- (3) $q \circ \Delta : A \rightarrow A^{\mathcal{d}\mathcal{U}}/I$ is injective, where $q : A^{\mathcal{d}\mathcal{U}} \rightarrow A^{\mathcal{d}\mathcal{U}}/I$ is the canonical quotient map.
- (4) $(q \circ \Delta)(A)$ is an essential ideal in $A^{\mathcal{d}\mathcal{U}}/I$.

Proof. For (1), suppose $(f_n)^\bullet, (g_n)^\bullet \in I$, $(h_n) \in A^{\mathcal{U}}$, and $\lambda \in \mathbb{C}$. Suppose that $f_n|_{B(r_n)}, g_n|_{B(s_n)} \equiv 0$, where $\lim_{\mathcal{U}} r_n = \lim_{\mathcal{U}} s_n = 0$. Then

$$\lambda f_n|_{B(r_n)}, (f_n + g_n)|_{B(\min(r_n, s_n))}, (f_n \cdot h_n)|_{B(r_n)} \equiv 0;$$

since $\lim_{\mathcal{U}} \min(r_n, s_n) = 0$, we have $\lambda f_n, f_n + g_n, f_n \cdot h_n \in I$ and I is an ideal.

We now prove that I is closed. Suppose that $((f_n^m)^\bullet : m \in \mathbb{N})$ is a sequence from I such that $\lim_m (f_n^m)^\bullet = (g_n)^\bullet$; we must show that $(g_n)^\bullet \in I$. Suppose that $f_n^m|_{B(r_n^m)} \equiv 0$ with $\lim_{n, \mathcal{U}} r_n^m = \infty$ for each m . Fix $k \in \mathbb{N}$

and take $m \in \mathbb{N}$ such that $\|(f_n^m)^\bullet - (g_n)^\bullet\| < \frac{1}{k}$. For \mathcal{U} -many n we have $\|f_n^m - g_n\| < \frac{1}{k}$ and $r_n^m \geq k$. Thus, if we set

$$X_k := \{n \in \mathbb{N} : n \geq k \text{ and } |g_n(t)| < \frac{1}{k} \text{ for } t \in B(k)\},$$

we have that $X_k \in \mathcal{U}$. For $n \in \mathbb{N}$, set $l(n) := \max\{k \in \mathbb{N} : n \in X_k\}$. Note that $n \in X_k$ implies that $l(n) \geq k$, whence $\lim_{n \in \mathcal{U}} l(n) = \infty$. Define $h_n := f_n \cdot \chi_{l(n)-1}$. Note that $(h_n)^\bullet \in I$; it remains to show that $(g_n)^\bullet = (h_n)^\bullet$. For $n \in \mathbb{N}$, we have $\|g_n - h_n\| \leq \sup_{t \in B(l(n))} |g_n(t)| \leq \frac{1}{l(n)}$, whence

$$\|(g_n)^\bullet - (h_n)^\bullet\| = \lim_{\mathcal{U}} \|g_n - h_n\| \leq \lim_{\mathcal{U}} \frac{1}{l(n)} = 0.$$

For (2), consider any sequence $(g_n) \in \ell^\infty(A)$ such that $g_n \equiv 1$ on $B(n)$. (For example, take $g_n := 1 - \chi_n$.) We claim that $q(g_n)^\bullet$ is an identity for the larger algebra $A^{\mathcal{U}}/I$. Indeed, consider arbitrary $q(f_n)^\bullet \in A^{\mathcal{U}}/I$. Then $f_n g_n - f_n$ vanishes on $B(n)$, whence $(f_n g_n - f_n)^\bullet \in I$ and $q(f_n g_n)^\bullet = q(f_n)^\bullet$.

For (3), suppose that $(q \circ \Delta)(f) = 0$. Then there is $(g_n)^\bullet \in I$ such that $\Delta(f) = (g_n)^\bullet$. Suppose that $g_n|_{B(r_n)} \equiv 0$ with $\lim_{\mathcal{U}} r_n = \infty$. Fix $t \in X$ and $\epsilon > 0$. Then for \mathcal{U} -many n , we have $\|f - g_n\| < \epsilon$ and $t \in B(r_n)$, whence $|f(t)| < \epsilon$. Since t and ϵ were arbitrary, we have that $f \equiv 0$.

We now prove (4). We first show that $(q \circ \Delta)(A)$ is an ideal in $A^{\mathcal{U}}/I$. Towards this end, fix $f \in A$ and $q((g_n)^\bullet) \in A^{\mathcal{U}}/I$; we must show that $q((f g_n)^\bullet) \in q(\Delta(A))$. In fact, we will show that $q((f g_n)^\bullet) = q(\Delta(f g_{\mathcal{U}}))$. Recall that

$$\|q((f g_n)^\bullet) - q(\Delta(f g_{\mathcal{U}}))\| = \inf_{\mathcal{U}} \lim \|f g_n - f g_{\mathcal{U}} - h_n\| : (h_n)^\bullet \in I.$$

Set $M := \sup_n \|g_n\|$. Fix $\epsilon > 0$. Fix $m \in \mathbb{N}$ such that $|f(t)| < \frac{\epsilon}{2M}$ when $t \in B(m)^c$. Let $\delta > 0$ witness the \mathcal{U} -equicontinuity of (g_n) on $B(m)$ with respect to $\frac{\epsilon}{3\|f\|}$ and fix a finite δ -net $\{t_1, \dots, t_k\}$ for $B(m)$. Fix $U \in \mathcal{U}$ such that $\{k \in \mathbb{N} : k \geq m\} \subseteq U$ and $|g_n(t_i) - g_{\mathcal{U}}(t_i)| < \frac{\epsilon}{3\|f\|}$ for $i = 1, \dots, k$ and $n \in U$. For $n \in U$, define $h_n \in A$ by $h_n := (f g_n - f g_{\mathcal{U}}) \chi_n$. (Define $h_n \in A$ for $n \notin U$ in an arbitrary fashion). It suffices to show that $\lim_{\mathcal{U}} \|f g_n - f g_{\mathcal{U}} - h_n\| \leq \epsilon$. Suppose $n \in U$. First consider $t \in B(m)$. Then $|f g_n(t) - f g_{\mathcal{U}}(t) - h_n(t)| = |f g_n(t) - f g_{\mathcal{U}}(t)|$. Take i such that $d(t, t_i) < \delta$. Then, for \mathcal{U} -many n , we have

$$|g_n(t) - g_{\mathcal{U}}(t)| \leq |g_n(t) - g_n(t_i)| + |g_n(t_i) - g_{\mathcal{U}}(t_i)| + |g_{\mathcal{U}}(t_i) - g_{\mathcal{U}}(t)| \leq \frac{\epsilon}{\|f\|},$$

whence $|f g_n(t) - f g_{\mathcal{U}}(t)| \leq \epsilon$. Now suppose that $t \in B(m)^c \cap B(n+1)$. Then $|f g_n(t) - f g_{\mathcal{U}}(t) - h_n(t)| \leq |f g_n(t) - f g_{\mathcal{U}}(t)| < \epsilon$ by choice of m . If $t \in B(n+1)^c$, then $f g_n(t) - f g_{\mathcal{U}}(t) - h_n(t) = 0$. It follows that $\lim_{\mathcal{U}} \|f g_n - f g_{\mathcal{U}} - h_n\| \leq \epsilon$, finishing the proof that $(q \circ \Delta)(A)$ is an ideal in $A^{\mathcal{U}}/I$.

We next show that $(q \circ \Delta)(A)$ is an essential ideal in $A^{\mathcal{U}}/I$. Suppose that $q(f_n)^\bullet \in A^{\mathcal{U}}/I$ is such that $q(f_n)^\bullet \cdot q(a)^\bullet = 0$ for all $a \in A$; we must show that $q(f_n)^\bullet = 0$.

Fix $t \in X$. Fix $a \in A$ such that $a(t) = 1$. Then there is $(g_n)^\bullet \in I$ such that $\lim_{\mathcal{U}} \|f_n a - g_n\| = 0$. For \mathcal{U} -most n , we have $t \in B(r_n)$, where g_n vanishes on $B(r_n)$. It thus follows that

$$\lim_{\mathcal{U}} |f_n(t)| \leq \lim_{\mathcal{U}} \|f_n a - g_n\| = 0.$$

Set

$$U_k := \{n \in \mathbb{N} : n \geq k \text{ and } |f_n(t)| \leq \frac{1}{k} \text{ for } t \in B(k)\}.$$

We claim that $U_k \in \mathcal{U}$. Fix $\delta > 0$ that witnesses \mathcal{U} -equicontinuity of (f_n) on $B(k)$ with respect to $\frac{1}{2k}$. Fix a finite δ -net F for $B(k)$. Then for \mathcal{U} -most n , $|f_n(t)| \leq \frac{1}{2k}$ for $t \in F$. Thus, given any $s \in B(k)$ and taking $t \in F$ such that $d(s, t) < \delta$, we have that $|f_n(s)| \leq |f_n(s) - f_n(t)| + |f_n(t)| \leq \frac{1}{k}$ for \mathcal{U} -most n .

For $n \in \mathbb{N}$, set $l(n) := \max\{k \in \mathbb{N} : n \in U_k\}$. For $n \in U_k$, we have $l(n) \geq k$, whence $\lim_{\mathcal{U}} l(n) = \infty$. Define $h_n \in A$ by $h_n = f_n \cdot \chi_{l(n)-1}$. As above, we have that $(h_n)^\bullet \in I$ and $\|f_n - g_n\| \leq \frac{1}{l(n)}$ whence $\lim_{\mathcal{U}} \|f_n - g_n\| \leq \lim_{\mathcal{U}} \frac{1}{l(n)} = 0$. \square

Since $q(\Delta(A))$ is an essential ideal in the unital C^* -algebra $A^{\mathcal{A}}/I$, we see that $\Sigma(A^{\mathcal{A}}/I)$ is a compactification of X , where $\Sigma(A^{\mathcal{A}}/I)$ denotes the Gelfand spectrum of $A^{\mathcal{A}}/I$. It turns out that this compactification is indeed the Stone-Cech compactification of X . Recall that $C_b(X)$ denotes the unital C^* -algebra of bounded, continuous, complex-valued functions on X and is naturally isomorphic to $C(\beta X)$, where βX denotes the Stone-Cech compactification of X .

Proposition 3.5. *There is an isomorphism $\Phi : A^{\mathcal{A}}/I \rightarrow C_b(X)$ such that $\Phi(q(\Delta(a))) = a$ for all $a \in A$.*

Proof. Define $\Phi : A^{\mathcal{A}} \rightarrow C_b(X)$ by $\Phi((f_n)^\bullet) := f_{\mathcal{U}}$. It is clear that Φ is a $*$ -morphism. We next observe that Φ is onto. Indeed, given $f \in C_b(X)$ and $n > 0$, define $f_n \in C_0(T)$ by $f_n = (1 - \chi_n)f$. Since f is bounded, we have that $(f_n) \in \ell^\infty(A)$. Since X is proper, f is uniformly continuous on bounded sets, whence (f_n) is \mathcal{U} -equicontinuous on bounded sets, that is, $(f_n)^\bullet \in A^{\mathcal{A}}$. It is clear that $\Phi((f_n)^\bullet) = f$.

Now suppose that $(f_n)^\bullet \in I$. Then by the definition of I , we have that $\Phi((f_n)^\bullet) = 0$, so Φ induces a surjection $\Phi : A^{\mathcal{A}}/I \rightarrow C_b(X)$. Suppose now that $\Phi((f_n)^\bullet) = 0$. For each $n > 0$, define a function $g_n \in A$ by $g_n = f_n \chi_{n-1}$. It is clear that $(g_n)^\bullet \in I$. Since $\lim_{\mathcal{U}} f_n(t) = 0$ for all $t \in X$ and $\|f_n - g_n\| \leq \max_{t \in B(o, n)} |f_n(t)|$, it follows that $(f_n)^\bullet = (g_n)^\bullet$, whence $(f_n)^\bullet \in I$, thus proving that $\Phi : A^{\mathcal{A}}/I \rightarrow C_b(X)$ is an isomorphism.

Finally, it is clear from the definition of Φ that $\Phi(q(\Delta(a))) = a$ for all $a \in A$. \square

From now on, we set $B := (A^{\mathcal{A}}/I)/((q \circ \Delta)(A))$, a unital C^* -algebra, and let $r : A^{\mathcal{A}} \rightarrow B$ denote the composition of q with the quotient map

$A^{\mathcal{A}}/I \rightarrow B$. Note that by the previous proposition, $B \cong C(X^*)$, where X^* denotes the *Stone-Cech remainder* $\beta X \setminus X$ of X .

We now introduce a group action into the picture:

Lemma 3.6. *Suppose that Γ acts isometrically on X .*

- (1) *The induced action of Γ on A further induces an action of Γ on $A^{\mathcal{A}}$ by $\gamma \cdot (f_n)^{\bullet} := (\gamma \cdot f_n)^{\bullet}$.*
- (2) *Both $A^{\mathcal{A}}$ and I are invariant under the action of Γ on $A^{\mathcal{A}}$ from (1).*

Proof. For (1), we need to verify that, for $(f_n), (g_n) \in \ell^{\infty}(A)$, if $\lim_{\mathcal{U}} \|f_n - g_n\| = 0$, then $\lim_{\mathcal{U}} \|\gamma \cdot f_n - \gamma \cdot g_n\| = 0$. However, this follows from the easy check that $\|\gamma \cdot f_n - \gamma \cdot g_n\| = \|f_n - g_n\|$ for each n .

We now prove (2). The fact that $A^{\mathcal{A}}$ is invariant under the action of Γ follows from the fact that Γ acts by isometries and thus takes bounded sets to bounded sets. We now prove that I is invariant under the action of Γ . Consider $\gamma \in \Gamma$ and $(f_n)^{\bullet} \in I$; we must show that $(\gamma \cdot f_n)^{\bullet} \in I$. Suppose that $f_n|_{B(o, r_n)} \equiv 0$ where $\lim_{\mathcal{U}} r_n = \infty$. Set $k := d(\gamma^{-1} \cdot o, o)$. Then for $r_n > k$, we have that $(\gamma \cdot f_n)|_{B(o, r_n - k)} \equiv 0$: if $t \in B(o, r_n - k)$, then

$$d(\gamma^{-1}t, o) \leq d(\gamma^{-1}t, \gamma^{-1}o) + d(\gamma^{-1}o, o) = d(t, o) + k \leq r_n,$$

whence $f_n(\gamma^{-1}t) = 0$. Since $r_n > k$ for \mathcal{U} -most n and $\lim_{\mathcal{U}} r_n - k = \infty$, it follows that $(\gamma \cdot f_n)^{\bullet} \in I$. \square

By the previous lemma, we have an induced action of Γ on $A^{\mathcal{A}}/I$ by setting $\gamma \cdot q(f_n)^{\bullet} := q(\gamma \cdot f_n)^{\bullet}$, whence we also get an action of Γ on B by setting $\gamma \cdot r(f_n)^{\bullet} := r(\gamma \cdot f_n)^{\bullet}$.

4. GROUPS ACTING PROPERLY AND ISOMETRICALLY ON A TREE

In this section, our locally compact space is simply a tree T given the usual path metric, namely $d(x, y)$ = the length of the shortest path connecting x and y . In this case, $A^{\mathcal{A}} = A^{\mathcal{U}}$. We further suppose that $\Gamma \curvearrowright T$ properly, isometrically and transitively. (Recall that the action is proper if the map $(g, t) \mapsto (gt, t) : G \times T \rightarrow T \times T$ is proper, meaning that inverse images of compact sets are compact.) In this case, $\text{Stab}(o)$ is finite, say of cardinality m . For a point $t \in T$, we let $x_{[o, t]}$ denote the geodesic segment connecting o and t .

Theorem 4.1. *If Γ acts properly, transitively, and isometrically on a simplicial tree T , then Γ is exact.*

Proof. For $t \in T$ and $i \in \mathbb{N}$, set

$$X(i, t) := \{\gamma \in \Gamma : \gamma \cdot o \in B(i) \text{ and } \gamma \cdot o \in x_{[o, t]}\}$$

and $x(i, t) = |X(i, t)|^{-1/2}$. Note that $x(i, t) = m \cdot \min(i, d(o, t))$. Define $T_i^{(n)} : \Gamma \rightarrow A$ by

$$T_i^{(n)}(\gamma)(t) = \begin{cases} x(i, t) & \text{if } t \in B(2n) \text{ and } \gamma \in X(i, t); \\ 0 & \text{otherwise.} \end{cases}$$

Now define $T_i : \Gamma \rightarrow B$ by $T_i(\gamma) := r((T_i^{(n)}(\gamma))^\bullet)$. We claim that these functions satisfy the criteria of Fact 2.2, whence the action of γ on X^* is amenable.

Certainly each $(T_i^{(n)}(\gamma))^\bullet$ is a positive element of $A^{\mathcal{U}}$; since r is a C^* -algebra homomorphism, we have that each $T_i(\gamma) \geq 0$ in B .

We now verify that $\langle T_i, T_i \rangle_B = 1_B$; in other words, we must show that $\sum_{\gamma \in \Gamma} T_i(\gamma)^2 = 1_B$. First observe that there is a finite $\Gamma_i \subseteq \Gamma$ such that $\sum_{\gamma \in \Gamma} T_i^{(n)}(\gamma)^2 = \sum_{\gamma \in \Gamma_i} T_i^{(n)}(\gamma)^2 = \chi_{B(2n)}$. Since $(\chi_{B(2n)})^\bullet + I$ is the unit of $A^{\mathcal{U}}/I$, it follows that $r((\chi_{B(2n)})^\bullet)$ is the identity of B . Now compute:

$$\begin{aligned}
\sum_{\gamma \in \Gamma} T_i(\gamma)^2 &= \sum_{\gamma \in \Gamma_i} T_i(\gamma)^2 \\
&= \sum_{\gamma \in \Gamma_i} (r((T_i^{(n)}(\gamma))^\bullet))^2 \\
&= \sum_{\gamma \in \Gamma_i} r(((T_i^{(n)}(\gamma))^\bullet)^2) \\
&= r\left(\sum_{\gamma \in \Gamma_i} ((T_i^{(n)}(\gamma))^\bullet)^2\right) \\
&= r\left(\sum_{\gamma \in \Gamma_i} ((T_i^{(n)}(\gamma))^2)^\bullet\right) \\
&= r\left(\left(\sum_{\gamma \in \Gamma_i} T_i^{(n)}(\gamma)^2\right)^\bullet\right) \\
&= r((\chi_{B(o, 2n)})^\bullet) \\
&= 1_B.
\end{aligned}$$

It remains to prove that, for each $\gamma_1 \in \Gamma$, we have $\lim_{i \rightarrow \infty} \|T_i - \delta_{\gamma_1} * T_i\|_2 = 0$. It is straightforward to compute that $\delta_{\gamma_1} * T_i = \gamma_1 \cdot T_i(\gamma_1^{-1}\gamma)$. It follows that $\|T_i - \delta_{\gamma_1} * T_i\|_2^2$ is equal to

$$\left\| \sum_{\gamma \in \Gamma} (T_i(\gamma)^2 + (\gamma_1 \cdot T_i(\gamma_1^{-1}\gamma))^2 - 2T_i(\gamma)\gamma_1 \cdot T_i(\gamma_1^{-1}\gamma)) \right\|_B. \quad (\dagger)$$

Now $\gamma_1 T_i^{(n)}(\gamma_1^{-1}\gamma)(t) = T_i^{(n)}(\gamma_1^{-1}\gamma)(\gamma_1^{-1}(t))$, which is only nonzero if:

- (1) $\gamma_1^{-1}t \in B(2n)$;
- (2) $\gamma_1^{-1}\gamma \cdot o \in B(i)$;
- (3) $\gamma_1^{-1}\gamma \cdot o \in x_{[o, \gamma_1^{-1}t]}$.

Also notice that $\sum_{\gamma \in \Gamma} (\gamma_1 \cdot T_i^{(n)}(\gamma_1^{-1}\gamma))^2 = \chi_{\gamma_1 \cdot B(2n)}$, so (\dagger) equals

$$\|r(\chi_{B(2n)})^\bullet + r(\chi_{\gamma_1 B(2n)})^\bullet - 2 \sum_{\gamma \in \Gamma} (T_i^{(n)}(\gamma)\gamma_1 \cdot T_i^{(n)}(\gamma_1^{-1}\gamma))^\bullet\|_B,$$

which in turn equals

$$\inf_{n, \mathcal{U}} \{ \lim \| \chi_{B(2n)} + \chi_{\gamma_1 \cdot B(2n)} - 2 \sum_{\gamma \in \Gamma} T_i^{(n)}(\gamma) \gamma_1 \cdot T_i^{(n)} - g_n - a \| \}, \quad (\dagger\dagger)$$

where $(g_n)^\bullet$ ranges over I and a ranges over A .

Set

$$a(t) = (\chi_{B(2n)} + \chi_{\gamma_1 \cdot B(2n)} - 2 \sum_{\gamma \in \Gamma} T_i^{(n)}(\gamma) \gamma_1 \cdot T_i^{(n)}) \cdot \chi_{B(i) \cup \gamma_1 \cdot B(i)}.$$

Set $g_n(t) = \chi_{B(2n) \triangle \gamma_1 \cdot B(2n)}$. Finally set

$$O(i, n) = (B(2n) \cap \gamma_1 \cdot B(2n)) \setminus (B(i) \cup \gamma_1 \cdot B(i)).$$

Then $a \in A$, $(g_n)^\bullet \in I$ (as $g_n|_{B(n)} \equiv 0$) and the value in $(\dagger\dagger)$ is bounded by

$$\lim_{n, \mathcal{U}} \sup_{t \in O(i, n)} |2 - 2 \sum_{\gamma \in \Gamma} T_i^{(n)}(\gamma) \gamma_1 \cdot T_i^{(n)}(t)|. \quad (\dagger\dagger\dagger)$$

Let $Z(i, t)$ be the set

$$\{ \gamma \in \Gamma : \gamma \cdot o \in B(i), \gamma \cdot o \in x_{[o, t]}, \gamma_1^{-1} \gamma \cdot o \in B(i), \gamma_1^{-1} \gamma \cdot o \in x_{[o, \gamma_1^{-1} t]} \}.$$

Set $k := d(\gamma_1 \cdot o, o)$. For n sufficiently large and for $t \in O(i, n)$, we have $|Z(i, t)| = i - k$ and $2 \sum_{\gamma \in \Gamma} (T_i^{(n)}(\gamma) \gamma_1 \cdot T_i^{(n)})(t) = \frac{i-k}{im}$, whence the quantity appearing in $(\dagger\dagger\dagger)$ equals $2 - 2\frac{i-k}{im}$, which goes to 0 as $i \rightarrow \infty$ as desired. \square

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